B.Com.(H)/ B.A.(H) Economics IV Semester
Generic Elective: Elements of Analysis
Paper code: 32355444
<u>Assignment III: Infinite Series</u>

Q1. Prove that a geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \; (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then show that the sum is $\frac{a}{1-r}$.

Q2. Prove that if the series $\sum u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.
Give examples of series $\sum u_n$ in each of the following cases:

a) $\lim_{n\to\infty} u_n = 0$ and $\sum u_n$ converges

b) $\lim_{n\to\infty} u_n = 0$ and $\sum u_n$ diverges

Q3. For the following series, determine whether the series converges or diverges, If it converges, find its sum:

a) $\sum_{n=1}^{\infty} \left(\frac{1}{n+3} - \frac{1}{n+4}\right)$

b) $\sum_{n=5}^{\infty} \left(\frac{e}{\pi}\right)^{n-1}$

c) $\sum_{n=1}^{\infty} \frac{4^{n+2}}{7^{n-1}}$

d) $\ln\left(1 - \frac{1}{4}\right) + \ln\left(1 - \frac{1}{9}\right) + \cdots + \ln\left(1 - \frac{1}{(k+1)^2}\right) + \cdots$

Q4. In each part, find all values of x for which the series converges, and find the sum of the series for those values of x:

a) $\frac{1}{x^2} + \frac{2}{x^3} + \frac{4}{x^4} + \frac{8}{x^5} + \cdots$

b) $e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \cdots$

Q5. In each part, show that the series diverges:

a) $\sum_{n=1}^{\infty} \frac{n^2+n+3}{2n^2+1}$

b) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

Q6. In each part, test if the series converges:

a) $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$

b) $\sum_{n=1}^{\infty} \frac{5\sin^2 n}{n!}$

c) $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^{17}}$

d) $\sum_{n=1}^{\infty} \frac{8}{5^n+1}$

e) $\sum_{n=1}^{\infty} \frac{4^n}{n^2}$

f) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

g) $\sum_{n=1}^{\infty} \left(\frac{3n+2}{2n-1}\right)^n$

h) $\sum_{n=1}^{\infty} (1-e^{-n})^n$

i) $\sum_{n=0}^{\infty} \frac{(n+4)!}{4!n!4^n}$

j) $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

Q7. In each part, test if the alternating series converges:

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{\sqrt{n}+1}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n}$

Q8. In each part, classify the series as absolutely convergent, conditionally convergent or divergent:

a) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+2}{n(n+3)}$

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{2n-1}}{n^2+1}$

SUMMARY OF CONVERGENCE TESTS
INFINITE SERIES

Sr. No.	Name	Statement	Comments
1	Sequence of Partial Sums Test	Let $\{s_n\}$ be the sequence of partial sums of the series $u_1 + u_2 + u_3 + \cdots$. If the sequence $\{s_n\}$ converges to a limit S, then the series is said to converge to S and if the sequence $\{s_n\}$ diverges, then the series is said to diverge. In case of a convergent series $\sum_{n=1}^{\infty} u_n = S$	Useful when s_n is in a closed form or $\sum u_n$ is a telescoping series.
2	Necessary condition for convergence or Divergence test	If $\sum u_n$ converges then $\lim_{n\to\infty} u_n = 0$	Useful for divergent series. Show $\lim_{n\to\infty} u_n \neq 0$. This would imply $\sum u_n$ diverges
3	Comparison Test	Let $\sum u_n$ and $\sum v_n$ be two series with non negative terms such that $u_n \leq v_n \forall n$. Then if $\sum v_n$ converges $\Rightarrow \sum u_n$ converges. & If $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges	Useful when there are terms in log, sine or cosine. Use geometric series or p-series for comparison. Unless specified, use it as a last resort.
4	Limit Comparison Test	Let $\sum u_n$ and $\sum v_n$ be two series with positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$. Then if l is finite and non-zero, both the series converge or diverge together. If $l = 0$, then if $\sum v_n$ converges $\Rightarrow \sum u_n$ converges. If $l = \infty$, then if $\sum v_n$ diverges $\Rightarrow \sum u_n$ diverges	Useful when u_n is of the form $\frac{p(n)}{q(n)}$, where $p(n) = a_0 + a_1 n + \cdots + a_t n^t$ & $q(n) = b_0 + b_1 n + \cdots + b_k n^k$ choose $v_n = \frac{n^t}{n^k} = \frac{1}{n^{k-t}}$ which is a p-series.
5	Root Test	Let $\sum u_n$ be a series with positive terms such that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$. Then, (i) If $l < 1$, the series converges (ii) If $l > 1$, the series diverges (iii) If $l = 1$, the test is inconclusive	Useful when u_n involves n^{th} powers
6	Ratio Test	Let $\sum u_n$ be a series with positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$. Then, (i) If $l < 1$, the series converges (ii) If $l > 1$, the series diverges (iii) If $l = 1$, the test is inconclusive	Useful when u_n involves factorials or terms of the type $x^n (x \in \mathbb{R})$

7	Raabe's Test	Let $\sum u_n$ be series with positive terms such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$. Then, (i) If $l < 1$, the series diverges (ii) If $l > 1$, the series converges (iii) If $l = 1$, the test fails	Useful when Ratio Test fails
8	Alternating Series Test or Leibnitz Test	Let the series be of the form $\sum(-1)^n u_n$ or $\sum(-1)^{n+1} u_n$,where $u_n > 0$. Then the series converges if the following conditions hold: (i) $u_n \geq u_{n+1}\ \forall n$ (ii) $\lim_{n\to\infty} u_n = 0$	Used only for Alternating Series
9	Ratio Test for Absolute Convergence	Let $\sum u_n$ be a series with non-zero terms such that $\lim_{n\to\infty} \frac{\lvert u_{n+1}\rvert}{\lvert u_n\rvert} = l$. Then (i) If $l < 1$, the series converges absolutely (ii) If $l > 1$, the series diverges (iii) If $l = 1$, the test is inconclusive	The series need not have positive terms and need not be alternating to use this test.
10	Root Test for Absolute Convergence	Let $\sum u_n$ be a series such that $\lim \sqrt[n]{\lvert u_n\rvert} = l$. Then, (i) If $l < 1$, the series converges absolutely (ii) If $l > 1$, the series diverges (iii) If $l = 1$, the test is inconclusive	Useful when $\lvert u_n\rvert$ involves n^{th} powers
11	Cauchy Criterion of Convergence for series	Let $\sum u_n$ be a series. Then the series is convergent iff for every $\epsilon > 0$, $\exists\ M \in \mathbb{N}$, such that $\lvert s_n - s_m\rvert < \epsilon\ \ \forall n, m \geq M$	Here we need to check if the sequence of partial sums $< s_n >$ is Cauchy or not.